AD/A-003 892

PROPERTIES OF THE BIVARIATE DELAYED
POISSON PROCESS

A. J. Lawrence, et al

Naval Postgraduate School

Prepared for:

Office of Naval Research

July 1974

AD/A-003 892

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS-55Lw74071	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Properties of the Bivariate Delayed Poisson Process		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. J. Lawrence P. A. W. Lewis		8. CONTRACT OR GRANT NUMBER(s) NR042-284
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N; RR014-05-01; NR 042-284 WR 5-0019
11. CONTROLLING OFFICE NAME AND ADDRESS Chief of Naval Research Arlington, Virginia 22217		12. REPORT DATE July 1974
		13. NUMBER OF PAGES 28 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Bivariate Poisson point process
Random delays of Poisson events
Infinite Divisibility

Asynchronous counting
Semi-synchronous intervals

PRICES SUBJECT TO CHANGE

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The bivariate Poisson point process introduced in Cox and Lewis (1972), and there called the bivariate delayed Poisson process, is studied further; the process arises from pairs of delays on the events of a Poisson process. In particular, results are obtained for the stationary initial conditions, the joint distribution of the number of operative delays at an arbitrary time, the asynchronous counting distribution, and two semi-synchronous interval distributions. The joint delay distribution employed allows for
(cont. next page)

DD FORM 1473 EDITION OF 1 NOV 68 IS OBSOLETE
1 JAN 73
S/N 0102-014-6601

UNCLASSIFIED

(20)

dependence and two-sided delays, but the model retains the independence between different pairs of delays.

1. Introduction.

The bivariate delayed Poisson process is a particular type of bivariate point process, that is a point process with two types of event occurring along a time axis. Bivariate point processes were first studied by Cox and Lewis (1972) in their Berkeley Symposium paper and here their analysis of the bivariate Poisson processes (without Poisson noise) is carried further. This bivariate point process is formed by subjecting the events of a main Poisson process to independent pairs of delays; the two types of delay for each main event produce the two types of event in the bivariate point process. Since it is well known that a Poisson process whose events are independently and identically displaced remains a Poisson process, the events of each type considered separately form two Poisson processes. They cannot, however, be independent Poisson processes because their events are associated in pairs by the displacement centres (the events of the original Poisson process). The resulting bivariate point process is by definition a particular type of <u>bivariate Poisson process</u> because its marginal processes for events of each type are themselves (univariate) Poisson processes. Cox and Lewis (1972) assumed a model in which the delays within pairs were independent and non-negative, and discussed the transient and asynchronous bivariate counting processes. We shall mainly assume that the delays within pairs are dependent and two-sided and will first obtain the stationaly initial conditions. These lead to the joint distribution of the numbers of delays of each type in progress at an arbitrary time and a general form of the asynchronous bivariate counting distribution. Finally, we obtain the semi-synchronous marginal interval distribution, that is the distribution of the time from an event of one type to the next event of the opposite type; such distributions are of obvious use in the statistical analysis

of bivariate point processes, and have been studied in the context of the general theory by Wisniewski (1972). Milne (unpublished Ph.D. thesis), has shown that the bivariate Poisson process studied here is the most general form for an infinitely div 'ble bivariate point process.

2. Notation and Description of the Model.

The bivariate delayed Poisson process is constructed from a main Poisson process of rate μ. Associated with each of its events is a pair of delays Y_a and Y_b; thus an event at time t in the main Poisson process produces an event at time $t + Y_a$ in the a-process and an event at time $t + Y_b$ in the b-process. We will assume that the random variables Y_a and Y_b with ranges $(-\infty,\infty)$ have finite means and a dependent joint distribution without any atoms. The pairs (Y_a,Y_b) associated with different main events are assumed to be independently and identically distributed.

The random variables $N_a(t,t+t_a)$, $N_b(t,t+t_b)$ denote the numbers of type-a and type-b events in the intervals $(t,t+t_a]$ and $(t,t+t_b]$, respectively; when there can be no confusion these will be abbreviated to $N_a(t_a)$ and $N_b(t_b)$, with joint probability generating function (j.p.g.f.),$\varphi(z_1,z_2;t_a,t_b)$.

3. Stationary Initial Conditions.

The purpose and methodology of stationary initial conditions for univariate point processes have been described in Lawrance (1972); they carry over quite obviously to bivariate processes and so will not be laboured here. We wish to study the behaviour of the process at an arbitrary time, and this is defined as the time t as $t \to \infty$. It is apparent for the bivariate delayed Poisson process that at any time t there may be a (random) number of events of each type in the course of being delayed. The initial conditions must

2

therefore specify jointly the distribution of these numbers and the individual

residual delay times measured from t. We begin by dealing with the numbers

and delay times which are associated with main events occurring before time

t; however, when delays can oe negative (or advances), we have also to consider

delays associated with main events after time t.

The behaviour at time t yields to an analysis of the pairwise behav-

iour of (Y_a, Y_b) for each main event in $(0, t]$. There are four possibilities

concerning the type-a and type-b events which terminate Y_a and Y_b: both

occur after time t, type-a occurs after time t and type-b before time t,

vice-versa, and both occur before time t. Let there be n main events in

$(0, t]$, with $Z_t, Z_t^{(a)}, Z_t^{(b)}$ and $n - Z_t - Z_t^{(a)} - Z_t^{(b)}$ main events giving pairs

of type-a and type-b events of the above four types, respectively. Let

$\{Y_a(i), Y_b(i)\}$ for $i = 1, 2, \ldots, Z_t$ be the delay times past t associated

with main events counted by Z_t, and similarly let $Y_a'(i)$ for $i = 1, 2, \ldots, Z_t^{(a)}$

and $Y_b'(i)$ for $i = 1, 2, \ldots, Z_t^{(b)}$ be the delay times past t asociated with

the other two situations.

The key feature of the present model which makes it tractable is that,

given the n main events in $(0, t]$, they are independent and uniformly dis-

tributed over this interval. Thus we can see that

$$
P \begin{bmatrix}
Z_t = k, \ Y_a(i) \geq a_i, \ Y_b(i) \geq b_i \quad \text{for} \quad i = 1, 2, \ldots, k \\
Z_t^{(a)} = k_a, \ Y_a'(i) \geq a_i' \quad \text{for} \quad i = 1, 2, \ldots, k_a \\
Z_t^{(b)} = k_b, \ Y_b'(i) \geq b_i' \quad \text{for} \quad i = 1, 2, \ldots, k_b
\end{bmatrix}
$$

$$
= \sum_{n=k+k_a+k_b}^{\infty} e^{-\mu t} \frac{(\mu t)^n}{n!} \ \frac{n!}{(n-k-k_a-k_b)! \ k! \ k_a! \ k_b!}
$$

3

$$\times \left\{ \frac{1}{t} \int_{v=0}^{c} P(Y_a<v, Y_b<v)dv \right\}^{n-k-k_a-k_b}$$

$$\times \prod_{i=1}^{k} \left\{ \frac{1}{t} \int_{v=0}^{t} P(Y_a>v+a_i, Y_b>v+b_i)dv \right\}$$

$$\times \prod_{i=1}^{k_a} \left\{ \frac{1}{t} \int_{v=0}^{t} P(Y_a>v+a_i', Y_b<v) \, dv \right\}$$

$$\times \prod_{i=1}^{k_b} \left\{ \frac{1}{t} \int_{v=0}^{t} P(Y_a<v, Y_b>v+b_i')dv \right\} . \tag{3.1}$$

In (3.1) the variable v is used to denote that the typical main event being considered is located at time $t - v$, or a time point v before t. Letting $t \to \infty$ in (3.1) we obtain after a little simplification, and dropping the suffix t, the arbitrary time initial conditions as

$$P \begin{bmatrix} Z = k, Y_a(i) \ge a_i, Y_b(i) \ge b_i & \text{for} & i = 1,2,\ldots,k \\ Z^{(a)} = k_a, Y_a'(i) \ge a_i' & \text{for} & i = 1,2,\ldots,k_a \\ Z^{(b)} = k_b, Y_b'(i) \ge b_i' & \text{for} & i = 1,2,\ldots,k_b \end{bmatrix}$$

$$= \exp\left\{ -\mu \int_{v=0}^{\infty} [1-P(Y_a<v, Y_b<v)]dv \right\}$$

$$\times (k!)^{-1} \prod_{k=1}^{k} \left[\mu \int_{v=0}^{\infty} P(Y_a>v+a_i, Y_b>v+b_i)dv \right]$$

$$\times (k_a!)^{-1} \prod_{i=1}^{k_a} \left[\mu \int_{v=0}^{\infty} P(Y_a>v+a_i', Y_b<v)dv \right]$$

$$\times (k_b!)^{-1} \prod_{i=1}^{k_b} \left[\mu \int_{v=0}^{\infty} P(Y_a<v, Y_b>v+b_i')dv \right] . \tag{3.2}$$

4

Both (3.1) and (3.2) are valid for $k, k_a, k_b \geq 0$ subject to the interpretation that when any of k, k_a, k_b are zero the corresponding products are omitted.

We next consider the behaviour at any particular time, which we take as zero, which is due to negative delays. We must consider the behaviour at 0 due to main events in $(0,t]$ and then let $t \to \infty$. The situation is similar to that just considered except that we are interested in the 'overshoots' of the delays past zero in the negative direction. For each event in $(0,t]$ we either have a double overshoot, an overshoot of type-a only, an overshoot of type-b only, or no overshoot at all. Let these overshoots, measured as positive quantities, be denoted by $X_a(i)$, $X_b(i)$ for $i = 1,2,\ldots,k$, $X_a'(i)$ for $i = 1,2,\ldots,k_a$ and $X_b'(i)$ for $i = 1,2,\ldots,k_b$. Here we have changed from Y to X in our notation of the 'forward' initial conditions, and will use ξ instead of Z for counting the main events. The 'backward' initial conditions can now be obtained analogously to the forward ones, and yield after some reduction

$$
P \left[\begin{array}{l} \xi \quad = k, \; X_a(i) \geq a_i, \; X_b(i) \geq b_i \quad \text{for} \quad i = 1,2,\ldots,k \\[2mm] \xi^{(a)} = k_a, \; X_a'(i) \geq a_i' \quad \text{for} \quad i = 1,2,\ldots,k_a \\[2mm] \xi^{(b)} = k_b, \; X_b'(i) \geq b_i' \quad \text{for} \quad i = 1,2,\ldots,k_b \end{array} \right]
$$

$$
= \exp\{- \mu \int_{v=-\infty}^{0} [1-P(Y_a>v, \; Y_b>v)]dv\}
$$

$$
\times (k!)^{-1} \prod_{i=1}^{k} [\mu \int_{v=-\infty}^{0} P(Y_a<v-a_i, \; Y_b<v-b_i)dv]
$$

$$
\times (k_a!)^{-1} \prod_{i=1}^{k_a} [\mu \int_{v=-\infty}^{0} P(Y_a>v-a_i', \; Y_b<v)dv]
$$

$$
\times (k_b!)^{-1} \prod_{i=1}^{k_b} [\mu \int_{v=-\infty}^{0} P(Y_a>v, \; Y_b<v-b_i')dv] . \tag{3.3}
$$

5

A point to notice about both the forward and backward initial conditions is that they do not simplify appreciably when the delays within pairs are independent. Even if Y_a and Y_b are independent in the model, the initial pairwise delays, $X_a(i)$ and $X_b(i)$, if there are any, will be dependent.

4. The Numbers of Events in the Course of Being Delayed.

One aspect of the bivariate delayed Poisson process, considered at an arbitrary time t, is the number of events in the main process generated before time t which are delayed until after time t, and conversely the number of events generated after time t which are advanced to before time t. Now the bivariate delayed Poisson process can be regarded as a simple bivariate case of Lewis's branching Poisson process (Lewis, 1964), where a main event generates two subsidiary processes each of one event. In this terminology we will be obtaining the joint distribution of the numbers of each type of subsidiary process running at an arbitrary time. It is shown in Lewis (1964) for positive independent delays that marginally the distribution will be Poisson with means $E(Y_a)$, $E(Y_b)$; and also, for positive independent delays, the joint Poisson distribution is stated in Cox and Lewis (1972). The result of two-sided dependent delays is derived here, and perhaps places the earlier results in a wider perspective.

The required joint distribution is obtained from the forward and backward initial conditions by first setting the a_1, a_1' and b_1, b_1' in (3.2) and (3.3) equal zero. The backward and forward contributions will be independent because of the main Poisson process, and we require the joint distribution of $\{z+z^{(a)}, z+z^{(b)}\} + \{\xi+\xi^{(a)}, \xi+\xi^{(b)}\}$. Let the probability generating functions (p.g.f.'s) of the two pairs be $\varphi_1(z_1, z_2)$ and $\varphi_2(z_1, z_2)$. From (3.2) we thus have

6

$$\varphi_1(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{k_a=0}^{\infty} \sum_{k_b=0}^{\infty} \exp\left\{ - \int_{v=0}^{\infty} [1 - P(Y_a < v, Y_b < v)] dv \right.$$

$$\times (k!)^{-1} \left\{ \mu \, z_1 z_2 \int_{v=0}^{\infty} P(Y_a > v, Y_b > v) dv \right\}^k$$

$$\times (k_a!)^{-1} \left\{ \mu \, z_1 \int_{v=0}^{\infty} P(Y_a > v, Y_b < v) dv \right\}^{k_a}$$

$$\times (k_b!)^{-1} \left\{ \mu \, z_2 \int_{v=0}^{\infty} P(Y_a < v, Y_b > v) dv \right\}^{k_b}$$

$$= \exp\left\{ \mu(z_1 - 1) \int_{v=0}^{\infty} P(Y_a > v) dv + (z_2 - 1)\mu \int_{v=0}^{\infty} P(Y_b > v) dv \right.$$

$$\left. + \mu(z_1 - 1)(z_2 - 1) \int_{v=0}^{\infty} P(Y_a > v, Y_b > v) dv \right\} . \qquad (4.1)$$

This is the p.g.f. of a bivariate Poisson distribution λ and is essentially the result stated by Cox and Lewis (1972, equation (4.34)). For the negative delays (or advances) we have from (3.3)

$$\varphi_2(z_1, z_2) = \exp\left\{ \mu(z_1 - 1) \int_{v=-\infty}^{0} P(Y_a < v) dv + \mu(z_2 - 1) \int_{v=-\infty}^{0} P(Y_b < v) dv \right.$$

$$\left. + \mu(z_1 - 1)(z_2 - 1) \int_{v=-\infty}^{0} P(Y_a < v, Y_b < v) dv \right\} . \qquad (4.2)$$

The product of (4.1) and (4.2) gives $\psi(z_1, z_2)$, the required p.g.f., and remains of the well known infinitely divisible bivariate Poisson form. The sum of the first integrals in (4.1) and (4.2) shows that the marginal means are $E\{|Y_a|\}$ and $E\{|Y_b|\}$, respectively. We now show that the covariance term

$$\mu\left\{\int_{v=-\infty}^{0} P(Y_a < v,\ Y_b < v)dv + \int_{v=0}^{\infty} P(Y_a > v,\ Y_b > v)dv\right\}, \qquad (4.3)$$

is, apart from the factor μ, the expectation of $M(Y_a, Y_b)$ where

$$M(Y_a, Y_b) = \begin{cases} 0 & \text{for } Y_a Y_b \leq 0 \\ \\ \text{Min}\{|Y_a|,\ |Y_b|\} & \text{for } Y_a Y_b > 0. \end{cases} \qquad (4.4)$$

This may be seen from the successive equalities

$$E\{M(Y_a, Y_b)\} = E\{\min(Y_a, Y_b) | Y_a > 0,\ Y_b > 0\} P(Y_a > 0, Y_b > 0)$$

$$+ E\{\min(-Y_a, -Y_b) | Y_a < 0,\ Y_b < 0\} P(Y_a < 0,\ Y_b < 0)$$

$$= \int_{v=0} P(Y_a > v,\ Y_n > v | Y_a > 0,\ Y_b > 0)dv \times P(Y_a > 0,\ Y_b > 0)$$

$$+ \int_{v=0} P(-Y_a > v,\ -Y_b > v | Y_a < 0,\ Y_b < 0)dv \times P(Y_a < 0,\ Y_b < 0)$$

$$= \int_{v=0} P(Y_a > v,\ Y_b > v)dv + \int_{v=-\infty}^{0} P(Y_a < v,\ Y_b < v)dv. \qquad (4.5)$$

The interpretation of $M(Y_a, Y_b)$ is that it is the absolute length of the over-lap of a pair (Y_a, Y_b); if Y_a and Y_b are of opposite sign there is no overlap and $M(Y_a, Y_b)$ is zero. The joint distribution of the numbers of events which are in the course of being delayed at an arbitrary time thus has joint p.g.f.

$$\varphi(z_1, z_2) = \exp\{(z_1 - 1)\mu E(|Y_a|) + (z_2 - 1)\mu E(|Y_b|) + (z_1 - 1)(z_2 - 1)\mu E[M(Y_a, Y_b)]\}. \qquad (4.6)$$

When delays are non-negative, $|Y_a| = Y_a, |Y_b| = Y_b$, $M(Y_a Y_b) = \min(Y_a, Y_b)$ and (4.6) reduces the result stated by Cox and Lewis (1972). We may also note

8

that (4.6) applies to a bivariate generalized branching Poisson process, Lewis (1969), if Y_a and Y_b are understood as the durations of subsidiary processes. Further to Cox and Lewis's remarks in section 4.6 of their Berkeley paper, it is interesting to note that the initial conditions cannot be constructed from the joint probabilities in (4.6) and the marginal forward recurrence time distributions of Y_a and Y_b; this is because the dependency structure requires the joint distribution of these quantities.

5. The Stationary Bivariate Counting Process.

In this section we obtain the joint p.g.f. of $N_a(t_a)$ and $N_b(t_b)$, the number of events of type-a and type-b which occur in the intervals $(0, t_a]$ and $(0, t_b]$ of the stationary process. We know marginally that $N_a(t_a)$ and $N_b(t_b)$ are Poisson random variables of means μt_a and μt_b, so interest concerns their joint behaviour. Cox and Lewis (1972, Sections 4.3 and 4.4) obtained the joint distribution of $N_a(t_a)$ and $N_b(t_b)$ for pairwise independent and non-negative delays, and we present here the corresponding result for pairwise dependent and two-sided delays; this more general case is shown to produce a rather simpler result.

The type-a and type-b events in $(0, t]$ arise from three sources: (1) those from main events in $(-\infty, 0]$ which have been delayed into $(0, t_a]$ or $(0, t_b]$; (2) those from main events in $(0, \max(t_a, t_b)]$ which have been delayed or advanced and are still in $(0, t_a]$ or $(0, t_b]$; (3) those from main events in $(\max(t_a, t_b), \cdot]$ which have been advanced into $(0, t_a]$ or $(0, t_b]$. The contributions from these three sources are independently distributed in virtue of the Poisson main process, and we shall let their bivariate p.g.f's be denoted by $\varphi(z_1, z_2; -\infty, 0)$, $\varphi(z_1, z_2; 0, t_a, t_b)$ and $\varphi(z_1, z_2; t_a, t_b, \infty)$, respectively.

9

Derivations of all three are similar; particular cases of the first two appear in Cox and Lewis (1972), and so we limit ourselves to the derivation of $\varphi(z_1, z_2; t_a, t_b, \infty)$ which arises from negative delays. We first state for future reference,

$$\varphi(z_1, z_2; -\infty, 0) = \exp\{\mu(z_1-1) \int_{v=0}^{\infty} P(v<Y_a<v+t_a)dv + \mu(z_2-1) \int_{v=0}^{\infty} P(v<Y_b<v+t_b)dv$$

$$+ \mu(z_1-1)(z_2-1) \int_{v=0}^{\infty} P(v<Y_a<v+t_a, \ v<Y_b<v+t_b)dv\} \qquad (5.1)$$

and

$$\varphi(z_1, z_2; 0, t_a, t_b) = \exp\{\mu(z_1-1) \int_{v=0}^{\max(t_a,t_b)} P(-v<Y_a<t_a-v)dv + \mu(z_2-1) \int_{v=0}^{\max(t_a,t_b)} P(-v<Y_b<t_b-v)dv$$

$$+ \mu(z_1-1)(z_2-1) \int_{v=0}^{\max(t_a,t_b)} P(-v<Y_a<t_a-v, \ -v<Y_b<t_b-v)dv\}. \qquad (5.2)$$

For $\varphi(z_1, z_2; t_a, t_b, \infty)$ consider events which are advanced from $(\max(t_a,t_b), T]$ into $(0, t_a]$ or $(0, t_b]$, and ultimately let $T \to \infty$. For a main event at time $\max(t_a,t_b) + v$, its contribution to $\varphi(z_1, z_2; t_a, t_b, \infty)$ is

$$1 + (z_1-1)P(-v-t_0<Y_a<-v-t_1) + (z_2-1)P(-v-t_0<Y_b<-v-t_2)$$

$$+ (z_1-1)(z_2-1)P(-v-t_0<Y_a<-v-t_1, \ -v-t_0<Y_b<-v-t_2) \qquad (5.3)$$

where $t_0 = \max(t_a,t_b)$, $t_1 = \max(t_b-t_a,0)$, $t_2 = \max(t_a-t_b,0)$.

Conditional on the number of events in $(t_0, T]$, the terms (5.3) are independently and uniformly distributed over $(t_0, T]$. Hence, following the argument in section 3 to obtain the initial conditions, we obtain

10

$$\varphi(z_1,z_2;t_a,t_b,\infty) = \exp\{\mu(z_1-1)\int_{v=0}^{\infty} P(-v-t_0<Y_a<-v-t_1)dv + \mu(z_2-1)\int_{v=0}^{\infty} P(-v-t_0<Y_b<-v-t_2)dv$$

$$+ \mu(z_1-1)(z_2-1)\int_{v=0}^{\infty} P(-v-t_0<Y_a<-v-t_1, -v-t_0<Y_b<-v-t_2)dv\} . \tag{5.4}$$

We observe that (5.1), (5.2) and (5.4) are all of the infinitely divisible bivariate Poisson form, and thus it remains only to obtain expressions for the parameters in their product; this is the required bivariate p.g.f. Simple changes of variables in the integral coefficients of (z_1-1) show that their sum may be written

$$\mu \int_{v=-\infty}^{+\infty} P(v<Y_a<v+t_a)dv. \tag{5.5}$$

If we assume that Y_a has a density then this integral evaluates to μt_a straight away; generally we will make this assumption. However, if Y_a is restricted to non-negative values this may easily be proved without the assumption of a density. We thus have that $N_a(t_a)$ and $N_b(t_b)$ have marginal Poisson distributions of means μt_a and μt_b. Similarly, adding together the coefficients of $(z_1-1)(z_2-1)$ from (5.1), (5.2) and (5.4) gives

$$\mu \int_{v=-\infty}^{+\infty} P(v<Y_a<v+t_a, v<Y_b<v+t_b)dv \tag{5.6}$$

as $\text{Cov}\{N_b(t_a), N_b(t_b)\}$, a quantity which is always non-negative.

Thus, the general result for the bivariate p.g.f. of $N_a(t_a)$ and $N_b(t_b)$ is

$$\varphi(z_1,z_2;t_a,t_b) = \exp\{(z_1-1)\mu t_a + (z_2-1)\mu t_b$$

$$+ (z_1-1)(z_2-1)\mu \int_{v=-\infty}^{+\infty} P(v<Y_a<v+t_a, v<Y_b<v+t_b)dv\} . \tag{5.7}$$

11

When Y_a and Y_b are non-negative, the covariance becomes

$$\mu \int_{v=0}^{\min(t_a, t_b)} P\begin{bmatrix} 0 < Y_a < t_a - v \\ 0 < Y_b < t_b - v \end{bmatrix} dv + \mu \int_{v=0}^{\infty} P\begin{bmatrix} v < Y_a < v + t_a \\ v < Y_b < v + t_b \end{bmatrix} dv, \tag{5.8}$$

and this makes (5.7) correspond to Cox and Lewis (1972, equation (4.25)). Setting $z_1 = z_2 = 0$ in (5.7) gives the bivariate exponential forward recurrence times; further details are given by Cox and Lewis.

6. The Semi-Synchronous Interval Distributions.

Finally, in this paper we consider the distributions of intervals between events in the bivariate delayed Poisson process; the intervals between events of the same type have trivially, exponential distributions of parameter μ. Consequently, we shall be interested in the distribution of intervals between consecutive events of opposite type; these have been called by Cox and Lewis (1972) the semi-synchronous intervals. We take an arbitrary event of type-a, say, and require the distribution of the interval to the first subsequent event of type-b; this interval will be denoted by $X_a^{(b)}$. Some general theory relating synchronous, semi-synchronous and asynchronous interval distributions is given in Wisniewski (1972). We shall, however, develop our result from the definition in Cox and Lewis (1972),

$$P(X_a^{(b)} > x) = \lim_{t \downarrow 0} P\{N_b(t, t+x) = 0 \mid N_a(0, t) \geq 1\}, \tag{6.1}$$

which is analogous to the definition in Khintchine (1960, Section 9) of Palm functions for synchronous interval distributions. Taking (6.1) we write it as

$$P(X_a^{(b)} > x) = \lim_{t \downarrow 0} \left[\frac{P\{N_b(t, t+x) = 0\} - P\{N_a(0, t) = 0, N_b(t, t+x) = 0\}}{P\{N_a(0, t) \geq 1\}} \right] \tag{6.2}$$

12

Dividing both the numerator and denominator by t, and assuming that the individual limits exist, as will be evident in a moment for the present process, we have the general result

$$P(X_a^{(b)} > x) = -\mu^{-1} D_t^+ P\{N_a(0,t) = 0, N_b(t,t+x) = 0\}_{(t=0)}. \qquad (6.3)$$

Here D_t^+ denotes a right hand derivative and μ is the value of $t^{-1} P\{N_a(0,t) \geq 1\}$ as $t \downarrow 0$; the latter is trivially true from Khintchine's work since $N_a(0,t)$ counts events in a Poisson process of rate μ.

Our main concern now is thus to evaluate $P\{(N_a(0,t) = 0, N_b(t,t+x) = 0\}$ for the bivariate delayed Poisson process. Since $N_a(0,t)$ and $N_b(t,t+x)$ refer to the stationary process we must consider main events which have occurred in the intervals $(-T_1,0]$, $(0,t]$ and $(t,t+T_2]$, eventually letting T_1 and $T_2 \to \infty$. A main event which occurs at $-v$ in $(-T_1,0]$ does not lead to an a-event in $(0,t]$ or a b-event in $(t,t+x]$ when its delays are such that $Y_a \in [v,t+v]'$, $Y_b \in [t+v,t+v+x]'$, where prime denotes the complementary interval; similarly for a main event at v in $(0,t]$ the condition on the delays is $Y_a \in [-v,t-v]'$, $Y_b \in [t-v,t-v+x]'$, and finally for an event at $t+v$ in $(t,t+T_2]$ the condition is $Y_a \in [-(t+v),-v]'$, $Y_b \in [-v,x-v]'$. Use of the conditional property of the main Poisson process for each of the intervals $(-T_1,0]$, $(0,t]$ and $(t,t+T_2]$, leads to the result

$$P\{N_a(0,t) = 0, N_b(t,t+x) = 0\} = \exp\{\mu \int_{v=0}^{\infty} [-1 + P\begin{pmatrix} Y_a \in [v,t+v]' \\ Y_b \in [t+v,t+x]' \end{pmatrix}] dv$$

$$+ \mu \int_{v=0}^{t} [-1 + P\begin{pmatrix} Y_a \in [-v,t-v]' \\ Y_b \in [t-v,t-v+x]' \end{pmatrix}] dv + \mu \int_{v=0}^{+\infty} [-1 + P\begin{pmatrix} Y_a \in [-(t+v),-v]' \\ Y_b \in [-v,x-v]' \end{pmatrix}] dv\} . \qquad (6.4)$$

13

Slight changes of variable in the second and third terms then give

$$P\{N_a(0,t) = 0, \; N_b(t,t+x) = 0\} = \exp\{\mu \int_{u=-\infty}^{\infty} [-1+P\begin{pmatrix} Y_a \in [u,t+u]' \\ Y_b \in [t+u,t+u+x] \end{pmatrix}]du\} \; . \tag{6.5}$$

When the delays Y_a and Y_b are non-negative, the corresponding result is

$$P\{N_a(0,t) = 0, \; N_b(t,t+x) = 0\} = \exp\{\mu \int_{u=0}^{\infty} [-1+P\begin{pmatrix} Y_a \in [u,t+u]' \\ Y_b \in [t+u,t+u+x]' \end{pmatrix}]du$$
$$+ \mu \int_{u=0}^{t} [-1+P\begin{pmatrix} Y_a > u \\ Y_b \in [u,u+x]' \end{pmatrix}]du - \mu \int_{u=0}^{x} P(Y_b < u)du\} \; . \tag{6.6}$$

We can now use (6.3) for the semi-synchronous interval distribution and have

$$P\{X_a^{(b)} > x\} = \left[\!\!\left[D_t^+ \int_{u=-\infty}^{+\infty} [1-P\begin{pmatrix} Y_a \in [u,t+u]' \\ Y_b \in [u+u,t+u+x]' \end{pmatrix}]du \right]\!\!\right]_{(t=0)}$$
$$\times \exp\{\mu \int_{u=-\infty}^{+\infty} [-1+P\begin{pmatrix} Y_a \in [u,t+u]' \\ Y_b \in [t+u,t+u+x]' \end{pmatrix}]du\}_{(t=0)} . \tag{6.7}$$

The second term in (6.7) becomes $\exp(-\mu x)$ when $t = 0$ since the exponent is then of the form (5.5); the first term may be alternatively written as

$$D_t^+\left[\!\!\left[\int_{u=-\infty}^{+\infty} [P(u<Y_a<t+u) + P(t+u<Y_b<t+u+x) - P\begin{pmatrix} u<Y_a<t+u \\ t+u<Y_b<t+u+x \end{pmatrix}du \right]\!\!\right]_{(t=0)} \; ,$$

which then simplifies to

$$1 - D_t^+\left[\int_{u=-\infty}^{+\infty} P\begin{pmatrix} u<Y_a<t+u \\ t+u<Y_b<t+u+x \end{pmatrix}du \right]_{(t=0)} \; . \tag{6.8}$$

14

After writing the derivative in the incremental way and examining the expression at $t = 0$, it is found that (6.8) becomes $P(Y_b-Y_a [0,x]')$. Thus from (6.7), we have the main result of this section as

$$P\{X_a^{(b)} > x\} = P(Y_b-Y_a \in [0,x]')\exp(-\mu x),\qquad (6.9)$$

and it is clear that this is a proper survivor function. One particular case of (6.9) is when Y_a and Y_b have exponential distributions of parameters λ_a and λ_b; then the p.d.f. of $X_a^{(b)}$ has the mixed-exponential form

$$f_a^{(b)}(x) = \frac{\lambda_a}{\lambda_a+\lambda_b}(\mu+\lambda_b)e^{-(\mu+\lambda_b)x} + \frac{\lambda_b}{\lambda_a+\lambda_b}e^{-\mu x}.\qquad (6.10)$$

The distribution of $Y_b - Y_a$ appearing in (6.9) also features in other aspects of the bivariate delayed Poisson process, as mentioned by Cox and Lewis (1972), and the reason is fairly evident. For some purposes, the main Poisson process can be eliminated from the model by concentrating on (say) just the type-a events; a type-b event occurs at a distance $Y_b - Y_a$ from its type-a event, and these follow a Poisson process anyway. Such reasoning leads to a quicker derivation of (6.9), but our aim here has been to apply, for the first time it is believed, the semi-synchronous definition (6.1). It should be noted however, that the univariate process with displacements $Y_b - Y_a$ is not identical to our delayed Poisson process in all respects; for instance, in terms of initial conditions and numbers of events in the course of being delayed. Some joint counting distributions for the univariate delayed Poisson process have been obtained by Milne (1970) when pursuing identifiability questions.

Finally, we note that a joint distribution of interest is that of $X_a^{(b)}$ together with that of $X_a^{(a)}$, the interval from the arbitrary type-a event to

the next type-a event. We can define generally the required joint distribution
as

$$P\{X_a^{(a)}>x,\ X_a^{(b)}>y\} = \lim_{t\downarrow 0} P\{N_a(t,t+x) = 0,\ N_b(t,t+y) = 0 \mid N_a(0,t) \geq 1\}, \qquad (6.11)$$

or use the univariate delayed Poisson process with $Y \equiv Y_b - Y_a$ as the displacements. We follow the latter course. Taking our origin as an event in the Poisson process of type-a events, we require that there be no type-a events in $(0,x]$ and no type-b events in $(0,y]$. We consider type-a events which have been generated in $(-T_1,0]$ and $(x,T_2]$ and then let T_1 and T_2 tend to infinity; thus

$$P(X_a^{(a)}>x,\ X_a^{(b)}>y) =$$

$$\lim_{\substack{T_1\to\infty \\ T_2\to\infty}} \sum_{r=0}^{\infty} e^{-\mu T_1} \frac{(\mu T_1)^r}{r!} \left[\frac{1}{T_1} \int_{v=0}^{T_1} P(Y\in[v,v+y]')dv\right]^r \times e^{-\mu x} \times$$

$$\sum_{s=0}^{\infty} e^{-\mu T_2} \frac{(\mu T_2)^s}{s!} \left[\frac{1}{T_2} \int_{v=0}^{T_2} P(Y\in[-v-x,y-v-x]')dv\right]^s P(Y\in[0,y]') \qquad (6.12)$$

which may be simplified to

$$P(X_a^{(a)}>x\ X_a^{(b)}>y) = \exp\{-\mu x-\mu y+\mu \int_{u=-x}^{0} P(u<Y_b-Y_a<u+y)du\} \times P(Y_b-Y_a\in[0,y]'). \qquad (6.13)$$

This joint distribution has the marginal distributions already discussed. From a statistical analysis point of view it would be useful to obtain the correlation between $X_a^{(a)}$ and $X_a^{(b)}$; however, it appears from (6.13) that no simple general expression will be available, although (6.13) should yield the result with particular assumptions on the delay structure.

16

ACKNOWLEDGMENTS

REFERENCES

Cox, D. R. anc Lewis, P. A. W. (197.). Multivariate point processes. In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 3, (Lecam et al, ed.) pp. 401-448. Berkeley: Univ. of California Press.

Khintchine, A. Y. (1960). Mathematical Methods in the Theory of Queueing. London: Griffin.

Lawrance, A. J. (1972). Some models for stationary series of univariate events. In Stochastic Point Processes: Statistical Analysis, Theory and Applications (P. A. W. Lewis, ed.), pp. 199-256). New York: Wiley.

Lewis, P. A. W. (1964). A branching Poisson process model for the analysis of computer failure patterns. J. R. Statist. Soc. B 26, 398-456.

Lewis, P. A. W. (1969). Asymptotic properties and equilibrium conditions for branching Poisson processes. J. Appl. Prob. 6, 355-371.

Milne, R. K. (1970). Identifiability for random translations of Poisson processes. Z. Wahrscheinlichkeitstheorie 15, 195-201.

Wisniewski, T. K. M. (1972). Bivariate stationary point processes, fundamental relations and first recurrence times. Adv. Appl. Prob. 4, 296-317.